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A symmetric formulation for flux-limited convection schemes

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SUMMARY

The present paper introduces a new limiter formulation for the definition of bounded higher-order convection schemes within a finite-volume context. The new formulation emphasizes the symmetry property and brings into clearer view certain types of solution behaviour, in particular strong positive solution curvature. The new formulation can be shown graphically, in a diagram similar to the well-known Sweby diagram, and the various boundedness criteria in current use, in particular the total-variation diminishing (TVD) and positivity conditions, can be shown as regions in the new diagram. The formulation allows the definition of smooth limiters with simple and flexible functional forms, of which some example classes are given (along with the transformed versions of some existing limiters). The smooth classes can be extended to maintain positive solution behaviour on non-uniform grids in a simple and natural manner. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The use of non-linear convection schemes, which adjust themselves to achieve bounded solution behaviour, is now well established and a variety of such schemes have been proposed, in particular in the finite-volume context. Central to the definition of cell-centred finite-volume discretizations is the interpolation of *cell-face* values of the convected variable from the *cell-centre* values. This can be expressed in one dimension using the stencil as shown in Figure 1, in which a flow from left to right is assumed. For compactness, the solution slopes across the downwind and upwind faces of cell C are termed here a and b, respectively, see Figure 1.

The two principal methods in current use for the construction of non-linear scalar convection schemes are *limiter functions*, proposed by Sweby [1] and further developed by Roe [2], and *normalized variables*, proposed by Leonard [3], with further development by Gaskell and

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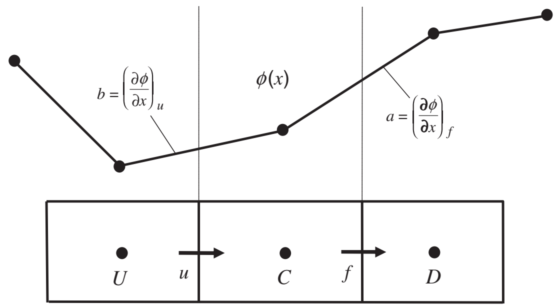

Figure 1. Upwind-biased stencil and notation.

Lau [4]. Each of these allows various boundedness criteria to be displayed graphically in associated diagrams, along with the various schemes attempting to meet the criteria. The relationships between these approaches were considered by Waterson and Deconinck [5] and a recent review of the subject can be found in [6]. Berger *et al.* [7] have proposed a new slope-limiter formulation using the same curvature-detection variable as Leonard.

The present paper proposes a new approach based on an alternative limiter-function formulation, which offers a different perspective and the possibility of defining more flexible and practical non-linear convection schemes. Consider first the limiter formulation proposed by Roe [2] in which the dependence of the value of the convected variable ϕ at face f on the neighbouring cell-centre values can be expressed as follows (in one dimension):

$$\phi_f = \phi_C + \tfrac{1}{2}\Delta x_C \Psi(r) b \tag{1}$$

where Δx_C is the width of cell C, $\Psi(r)$ is a limiter function and r is a gradient ratio:

$$r = \left(\frac{\partial\phi}{\partial x}\right)_f \Bigg/ \left(\frac{\partial\phi}{\partial x}\right)_u = \frac{a}{b} \tag{2}$$

One interpretation of the limiter function, $\Psi(r)$, is as a solution gradient non-dimensionalized by the upwind slope, b (some authors use the slope a instead and the inverse of r). The gradient ratio acts as a sensor for the state of the local solution, in particular detecting regions of strong solution curvature and extrema which may lead to unbounded solution behaviour. Limiter functions defining symmetric schemes, i.e. those giving equal weight to the slopes a and b in interpolating the face value, have the following property: $\Psi(r) = r\Psi(1/r)$.

Linear convection schemes based on the stencil considered, including the linearity-preserving 'κ-scheme' family [8], can be expressed using linear $\Psi(r)$ functions. However, the principal utility of the limiter function is as a mechanism for enforcing specific boundedness criteria, leading to various non-linear functional forms (mainly piecewise linear and polynomial ratios). The two most widely referenced criteria in this context are Harten's *total-variation diminishing* (TVD) condition [1, 9] and Spekreijse's more flexible *positivity* condition [10].

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2. PROPOSED FORMULATION

The new limiter-function formulation proposed here adopts alternative definitions of both the sensor variable and the limiter function itself. In place of the gradient ratio, r, defined above, the new sensor variable, s, is defined as follows:

$$s=\frac{a-b}{a+b}=\frac{r-1}{r+1} \tag{3}$$

i.e. as the ratio of the difference of the downwind and upwind slopes to their sum. In combination with this, a new limiter function, $\Theta(s)$, is defined, similar to that in (1) but now non-dimensionalizing the interpolation slope with the average of a and b:

$$\phi_f=\phi_C+\frac{1}{2}\Delta x_C\Theta(s)\left(\frac{a+b}{2}\right) \tag{4}$$

It can be seen that the variable s is closely related to the second derivative of the discrete solution and hence provides a measure of solution curvature. Any limiter function written in $\Psi(r)$ form can be transformed into $\Theta(s)$ form as follows:

$$\Theta(s)=(1-s)\Psi(r(s)) \tag{5}$$

in which $r(s)=(1+s)/(1-s)$.

The $\Theta(s)$ functions can be plotted in a diagram similar to Sweby's limiter diagram and boundedness criteria can be transformed into regions in the $\Theta(s)$ diagram. This is illustrated in Figures 2 and 3 showing, respectively, the transformed Sweby second-order TVD and Spekreijse positivity regions. Examination of the new formulation reveals a number of features:

- All linear schemes take the form of linear functions and appear as straight lines in the $\Theta(s)$ diagram, as shown in Figure 2. Examples are: first-order upwind (FOU), $\Theta(s)=0$; Fromm's scheme [11], $\Theta(s)=1$; central difference (CDS), $\Theta(s)=1+s$; linear upwind (LUI), $\Theta(s)=1-s$; and general κ-schemes, $\Theta(s)=1+\kappa s$.
- All symmetric schemes take the form of even $\Theta(s)$ functions (and all even $\Theta(s)$ functions are symmetric). As a result, the symmetry of a particular scheme is visually evident in the $\Theta(s)$ diagram, e.g. Fromm's scheme [11] ($\Theta(s)=1$), the only symmetric κ-scheme.
- The point $\Psi(1)=1$, defining linearity preservation, is transformed to $\Theta(0)=1$. This brings the smooth solution region, $s\approx 0$, to the centre of the diagram, with all monotonic solutions falling in the range $s\in[-1,1]$. The values $s=-1$ and 1 represent, respectively, the critical cases $a=0$ and $b=0$, the extreme limits of negative and positive solution curvature. Schemes passing through $(0,1)$ will respect linear solutions on arbitrary mesh distributions but they will only be formally second-order accurate on uniform meshes.
- All limiter functions attempting to fulfil the combined requirements of linearity preservation and either the TVD or positivity criteria must at least pass through the three critical points: $\Theta(0)=1$, $\Theta(-1)=0$ and $\Theta(1)=0$.
- All extrema are located in the region $|s|>1$ and symmetric extrema ($r=-1$) at $s=\infty$.
- As s is defined in terms of slopes, the form of the $\Theta(s)$ diagram and limiters defined in this form remain largely unchanged on non-uniform grids. One exception to this is the first-order downwind line which pivots about $s=-1$ for non-uniform grid distributions and represents one boundary of the TVD and positivity regions, see Figures 2 and 3.

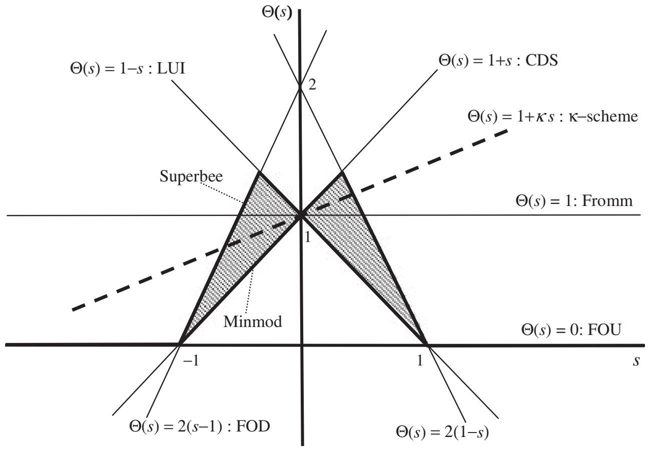

Figure 2. Sweby TVD region in $\Theta(s)$ form.

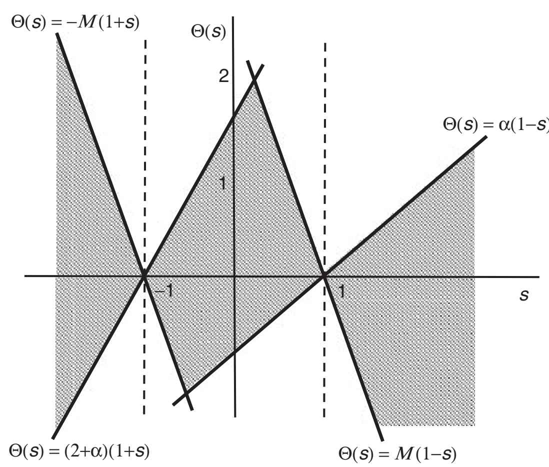

Figure 3. Spekreijse positivity region in $\Theta(s)$ form.

3. APPLICATION OF THE NEW FORMULATION

A number of well-known limited schemes immediately stand out in the new diagram. The upper and lower bounds of the second-order TVD region give the Superbee [12] and Minmod [13] limiters, respectively (as for $\Psi(r)$), see Figure 2. van Leer's MUSCL scheme [14] simply follows $\Theta(s)=1$ across the TVD region, while the van Leer Harmonic limiter [15] takes the form of a quadratic fit through the three critical points $(-1,0)$, $(0,1)$, and $(1,0)$: $\Theta(s)=1-s^2$. All of these schemes revert to FOU, $\Theta(s)=0$, for $|s|>1$. The smooth and continuous van Albada limiter [16] takes the form $\Theta(s)=(1-s^2)/(1+s^2)$. The $\Psi(r)$ and $\Theta(s)$ forms of these five 'classical' limiter functions are summarized in Table I.

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Table I. Five ‘classical’ limiter functions in $\Psi(r)$ and $\Theta(s)$ forms.

Name	References	$\Psi(r)$ form	$\Theta(s)$ form
Minmod	[13]	$\max[0, \min(r, 1)]$	$\max[0, \min(1-s, 1+s)]$
Superbee	[12]	$\max[0, \min(2r, 1), \min(r, 2)]$	$\max[0, \min\{1+s, 2(1-s)\},$ $\min\{1-s, 2(1+s)\}]$
MUSCL	[14]	$\max[0, \min\{2r, \frac{1}{2}(r+1), 2\}]$	$\max[0, \min\{2(1+s), 1, 2(1-s)\}]$
Harmonic	[15]	$\max[0, 2r/(r+1)]$	$\max[0, 1-s^2]$
van Albada	[16]	$r(r+1)/(r^2+1)$	$(1-s^2)/(1+s^2)$

With the exception of Minmod, none of the flux-limited schemes listed above are guaranteed to achieve bounded behaviour on non-uniform meshes [17] and one key advantage of the new formulation is in the definition of schemes that adjust themselves to overcome this limitation. This is quite straightforward to achieve for piecewise-linear limiter functions in both old and new forms, as illustrated for the general piecewise-linear class, termed GPL-κ in [6]:

$$\Theta(s) = \max[0, \min\{(\gamma+1)(1+s), 1+\kappa s, M(1-s)\}] \tag{6}$$

in which $M \geqslant 1$ controls the slope at $s=1$ and the ratio $\gamma = \Delta x_f / \Delta x_C$ is a measure of grid non-uniformity. This form of the scheme automatically adjusts itself to follow the first-order downwind line close to $s=-1$ and thereby maintains positive behaviour (assuming γ is constant for a given cell). This type of behaviour is much more difficult to achieve in the case of smooth limiter functions, exemplified by the van Leer Harmonic and van Albada schemes.

Consider a convenient new class of smooth, continuous schemes in $\Theta(s)$ form:

$$\Theta(s) = \frac{(1-s^2)}{1-\kappa s+\beta s^2} \tag{7}$$

These schemes are all tangential to a chosen κ-scheme at $s=0$, while the value of β controls the slope at $s=\pm 1$ and therefore the resolution of discontinuities: $\Theta'(\mp 1) = \pm 2/(1\pm\kappa+\beta)$. The class is smooth and continuous for $\beta > \kappa^2/4$ and for $\kappa=0$ it gives, respectively, the van Leer Harmonic [15] ($\beta=0$), van Albada [16] ($\beta=1$), and OSPRE [5, 6] ($\beta=\frac{1}{3}$) schemes. A necessary (and usually sufficient) condition to maintain positive behaviour on a non-uniform mesh is $\Theta'(-1) = 2/(1+\kappa+\beta) \leqslant (\gamma+1)$, which can be fulfilled by making β a function of γ. For example, $\kappa=0$ and $\beta=(1-\gamma)/(1+\gamma)$ give a version of the van Leer Harmonic limiter which is positive on non-uniform meshes (again assuming γ constant for a given cell). If desired for implementation this can now be converted back into $\Psi(r)$ form.

A further new class of schemes, which cannot be defined compactly in the $\Psi(r)$ form, consists of a simple quadratic fit through the monotonic region ($|s| \leqslant 1$):

$$\Theta(s) = \max[0, (1-s^2)(1+\kappa s+\beta s^2)] \tag{8}$$

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which is once again tangential to a κ-scheme at $s=0$ and with the value of β controlling the slopes at $s=\pm 1$. The slope at $s=-1$: $\Theta'(-1)=2(1-\kappa+\beta)$ and in the case of a non-uniform grid, as above, β can be expressed as a simple function of γ to maintain positivity.

4. CONCLUSIONS

A new limiter formulation has been proposed for the definition of bounded convection schemes within a finite-volume context. The new formulation emphasizes the symmetry property and brings into clearer view certain types of solution behaviour associated with loss of boundedness, in particular strong positive solution curvature. One advantage of the new formulation is the relatively simple functional forms that can be used to define smooth limiter functions and permit their simple and natural extension to non-uniform grids. Examples of both transformed versions of existing non-linear schemes and new limiter functions expressed in the proposed formulation have been presented. These include a new version of the van Leer Harmonic limiter which adjusts itself automatically to maintain positivity on non-uniform meshes. The detailed behaviour of these new limiters will be explored in a future publication.

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